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A Dynamical Approach to Anomalous Conductivity

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We study a process of anomalous diffusion of a variable resulting from the fluctuations of a dichotomous velocity whose two states, in the absence of perturbation, have the same waiting time distribution $\psi(t)$. In the long-time limit the function $\psi(t)$ is proportional to $t^{-\mu}$ with $2<\mu<3$. Previously this distribution along with the constraint on μ proved to be a dynamical realization of an α-stable Lévy process with $\alpha=\mu-1$. Here we study the response of this anomalous diffusion process to a perturbation which has the effect of truncating the inverse power law of one of the two states of the velocity for times $t>1/\varepsilon$, where ε is proportional to the intensity of the weak perturbation. We show that the resulting transport process is characterized by a succession of two regimes: the first still satisfies the prescriptions of the Green–Kubo approach to conductivity, and, in accordance with the nature of the anomalous diffusion studied here, corresponds to a state of increasing conductivity (IC); the second regime is characterized by a constant conductivity (CC). The transition from the IC to the CC regime takes place in a time of the order of $t\sim 1/\varepsilon$ and consequently the transition occurs at longer and longer times, as the perturbation intensity decreases. The final stationary regime corresponds to an asymmetric Lévy process of diffusion.

KEY WORDS: Anomalous conductivity; perturbation.

1. INTRODUCTION

The Green–Kubo method is the most widely used approach to determining the conductivity of a physical system from its microscopic dynamical properties. These relations are closely related to linear response theory

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and consequently to the foundations of ordinary statistical mechanics. To elucidate the important dynamical properties underlying ordinary statistical mechanics and the essentially equivalent Green–Kubo relations, we shall examine a model system which maintains the essential aspects of real processes and at the same time can be given a simple analytical treatment.

In the one-dimensional case a process of diffusion for a variable x can be described, if a dynamical approach is adopted,[1] by

$$\dot{x}(t) = \xi(t) \tag{1.1}$$

Here, if x is a space variable, ξ plays the role of a velocity with an erratic dependence on time. We assume that $\langle \xi \rangle = 0$, where the brackets denote an ensemble average over the velocity fluctuations. The properties of the diffusion process are essentially determined by the stationary correlation function

$$\Phi_{\xi}(t) = \frac{\langle \xi(0)\, \xi(t) \rangle_{\text{eq}}}{\langle \xi^2 \rangle_{\text{eq}}} \tag{1.2}$$

If the stationary process of diffusion is perturbed by the abrupt application of a constant perturbation, the response of the diffusing variable, according to the general prescription of linear response theory,[2] is described by

$$\frac{d}{dt} \langle x(t) \rangle = \lambda \int_0^t \langle \xi(0)\, \xi(t') \rangle_{\text{eq}}\, dt' \tag{1.3}$$

where λ is a suitable constant coefficient determined by the interaction between the diffusing system and, the external perturbation.

The traditional processes of unperturbed diffusion are characterized by the property that the time scale defined by

$$\tau = \int_0^\infty \Phi_{\xi}(t')\, dt' \tag{1.4}$$

is finite. This means that for times longer than τ the fluctuations of the variable x are uncorrelated, the central limit theorem holds, and with it ordinary Gaussian diffusion. The effect of a perturbation is straightforwardly predicted by adapting the same arguments as those used to derive the standard diffusion process to a system of reference moving with the velocity V given by

$$V = \lambda \langle \xi^2 \rangle_{\text{eq}} \int_0^\infty \Phi_{\xi}(t')\, dt' \tag{1.5}$$

This leads us to predict that after a time of the order of τ from the instant of abrupt application of the perturbation, the diffusing system reaches a new stationary condition given by a distribution function, still Gaussian, and translating with the velocity V.

The major aim of this paper is to discuss the same process of response to a perturbation in the case where the correlation function $\Phi_{\xi}(t)$ does not decay sufficiently fast to guarantee the conditions for ordinary diffusion. It has been shown by several authors[1,3–7] that it is possible to realize a dynamical process characterized by an inverse power-law autocorrelation function (1.2) with a deterministic prescription, such as either the Geisel *et al.* map (GNZM)[3] or the standard map (SM) in the accelerator state.[6] The autocorrelation function is then of the form

$$\lim_{t \to \infty} \Phi_{\xi}(t) \propto t^{-\beta} \tag{1.6}$$

with

$$0 < \beta < 1 \tag{1.7}$$

It has been shown[1,4] that the diffusion process corresponding to this condition is an α-stable Lévy process with index

$$\alpha = \beta + 1 \tag{1.8}$$

These theoretical results raise the question of what kind of response a dynamical system characterized by the properties (1.6) and (1.7) would have to a constant perturbation. The problem is of some interest because it implies a possible breakdown of the prescription (1.5), which is widely used in statistical mechanics to determine the conductivity following the application of a perturbation. We shall refer to (1.5) as the Green–Kubo (GK) prescription.

Does the GK prescription hold in the case of an inverse power-law autocorrelation function? It is evident that the GK cannot hold true in the form (1.5), since this would involve an infinite conductivity. On the other hand, the GK prediction might hold true, for short times, in the form (1.3). However, the resulting conductivity would steadily increase with time, until eventually it breaks the condition of weak response to the external perturbation, the condition on which the linear response treatment leading to (1.3) rests. It must also be pointed out that the derivation of (1.3) implies[2] the assumption that the equilibrium distribution of ξ is Maxwellian. We think that the former requirement is naturally fulfilled in the early part of the response process, regardless of the long-time behavior of the correlation function $\Phi_\xi(t)$. The fulfilment of the second one is made impossible by the dichotomic nature of the variable velocity in this case. This, however, would not prevent us from conveniently using the linear response prescription.[8] Furthermore, among the different ways of perturbing the diffusing system, we shall adopt one that not only fulfills the, spirit of the linear response theory, but also coincides formally with (1.3) in the early part of the response process.

The illustration of the process of perturbation needs a preliminary illustration of the nature of the dynamical system under study. As earlier said, the variable ξ fluctuates between only two values, $\xi = 1$ and $\xi = -1$. In the absence of perturbation, the two states are characterized by the same statistical weight and by the same distribution of sojourn times $\psi(t)$. The relation between $\Phi_\xi(t)$ and $\psi(t)$ can be established on the basis of renewal theory,[3]

$$\Phi_\xi(t) = \frac{1}{\langle t \rangle} \int_t^\infty (T-t)\, \psi(T)\, dT \tag{1.9}$$

which is exact under the assumption that ξ is a dichotomous variable taking opposite values. From this relation we immediately see that the form of the correlation function given by (1.6) is obtained if the waiting time distribution has the following behavior:

$$\lim_{t \to \infty} \psi(t) \propto \frac{1}{t^\mu} \tag{1.10}$$

In this specific case it is straightforward to prove that

$$\beta = \mu - 2 \tag{1.11}$$

With the use of numerical calculations it has been assessed[6, 7] that the standard map and other Hamiltonian systems in a condition of weak chaos, which is a phase space containing nonlinear resonant tori as well as a chaotic sea, realize (1.10) with

$$2 < \mu < 3 \tag{1.12}$$

This means that we have an α-stable Lévy process whose parameter α, according to (1.8), fulfills the condition $1 < \alpha < 2$.

We now assume that the effect of an external perturbation is that of changing the form of the distribution $\psi(t)$ of waiting times in the states $\xi = 1$ and $\xi = -1$. A bias is realized when the two distributions, which we denote by $\psi_1(t)$ and $\psi_{-1}(t)$, respectively, get different forms. A perturbation of the $\psi(t)$ affecting the region of short times would result in a trivial effect. Of much more significance is the case where one or both of the waiting-time distributions are perturbed in such a way as to lose the inverse-power-law nature. The form of the perturbation is chosen to ensure that the early-time region of the response fulfills the condition (1.3). The numerical results of an earlier paper[1] show that the system makes a transition from this GK regime to a final regime of constant conductivity. Herein we deepen our understanding of this final regime with the help of analysis to supplement our numerical arguments.

The outline of the paper is as follows. Section 2 is devoted to illustrating the dynamical model used for our numerical calculations. Section 3 contains a discussion of the conditions for stationary and nonstationary distributions Section 4 illustrates the theoretical predictions concerning how the shape of the probability distribution changes after switching on the perturbation. It is shown that the CC regime consists, in turn, of an earlier regime of shape readjustment and of a final, genuinely stationary regime corresponding to an asymmetric Lévy process. In Section 5 we make a comparison between numerical and theoretical results and approach the long-time regime where the theory applies starting from a short-time dynamical regime still dependent on the details of the generator of the inverse power law adopted. Concluding remarks are made in Section 6.

2. THE DYNAMICAL SYSTEM UNDER STUDY

The theoretical predictions of this paper refer to a region of long times where the details of the microscopic dynamics are lost and the behavior of the system is determined by the inverse power-law behavior (1.10) of the waiting time distribution. All the dynamical systems with the same inverse power law in the long-time regime are expected to fit these predictions. To explore the transition from a model-dependent, short-time regime to a long-time regime with a universal behavior, we need to make numerical calculations and adopt a specific dynamical model. For this purpose we make the simplest possible choice, namely, we consider the one-dimensional map introduced by Geisel *et al.*[3] in order to derive anomalous diffusion that evolves faster than normal.

Let us consider the unbiased case. Using the property of antisymmetry by reflection around $x = 0$ and the invariance by translation of a unit distance, we can express the map in the reduced range $0 \leqslant x \leqslant 1/2$ as

$$x_{n+1} = g(x_n) \tag{2.1}$$

with

$$g(x) = x + ax^z - 1, \qquad 0 \leqslant x \leqslant 1/2 \tag{2.2}$$

where the constant a is chosen to be $a = 2^z$.

The reduced map is obtained by iterating separately the motion within each unitary cell and the motion between cells. The coordinate x of the trajectory is decomposed into its integer part, the box number N, and the position y within a box ($x_n = N_n + y_n$), and so the reduced map is defined as[4]

$$\begin{aligned} y_{n+1} &= \tilde{g}(y), \qquad 0 < y < 1 \\ N_{n+1} &= \hat{g}(y_n) + N_n \end{aligned} \tag{2.3}$$

$\tilde{g}(y)$ is the reduced map for the reduced coordinate y illustrated in Fig. 1 and $\hat{g}(y)$ is used to change the box number N. The left and right branches are respectively associated with a decrement and an increment of one unit of N, and this completely determines the law $\tilde{g}(y)$. In practice, permanence in the left (right) laminar region corresponds to the velocity $\xi = dx/dt$ having the value $+1$, -1, respectively. The motion in regions I and J of Fig. 1 has a regular (laminar) character; it is easy to see that points corresponding to successive iterations are very near one another, so that particles remain in the laminar regions for a long time. This property depends of course on the fact that the reduced map $\tilde{g}(y)$ is tangent to the bisectrix $\tilde{g}(y) = y$ in $y = 0$ and $y = 1$. The motion in the laminar regions is interrupted by a chaotic motion in the interval K, which typically lasts for a small number of iterations; then the particle is again injected into I or J.

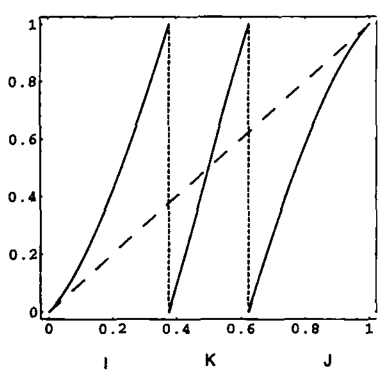

Fig. 1. The reduced map defined by Eq. (2.2), (2.3), with $z = 1.8$.

Neglecting the short time of permanence in the chaotic region K with respect to the much longer time of permanence in the laminar regions, we have that the dynamics of the map can be described as follows: the particle always moves with velocity of modulus one, changing direction at random after intervals of time which are typically long, but not fixed and follow a distribution $\psi(t)$. In practice, we have a system with a dichotomous velocity.

The statistical properties of the motion generated by the map (2.2) are determined by the waiting time distribution $\psi(t)$ in the laminar regions. The functions has an inverse power law behavior for long times and reads[4]

$$\psi(t) = \frac{(\mu - 1)\, B^{\mu - 1}}{(B + t)^{\mu}} \tag{2.4}$$

where μ is related to z in the map (2.2) by

$$\mu = \frac{z}{z - 1} \tag{2.5}$$

and

$$B = \frac{\mu - 1}{2} \tag{2.6}$$

We want to study a case of asymmetric diffusion, which means that the waiting time distributions $\psi_1(t)$ and $\psi_{-1}(t)$ relative to permanence in the right and left laminar regions, respectively, are different. The numerical experiment of applying the perturbation is made by replacing the deterministic law of the left laminar region (2.2) with

$$x_{n+1} = (1 + p)\, x_n + a x_n^z - 1 \tag{2.7}$$

p is the strength of the perturbation. The corresponding waiting time distribution is known and it is given by[4]

$$\psi_p(t) = \frac{2 \exp(pt/(\mu - 1))}{\{1 + (2/p)[\exp(pt/(\mu - 1)) - 1]\}^{\mu}} \tag{2.8}$$

Now, the important properties of the process discussed in this paper are not qualitatively altered if, instead of (2.8), we take the waiting time distribution of the left laminar region to be of the form

$$\psi_\varepsilon(t) = A_\varepsilon \frac{e^{-\varepsilon t}}{(B + t)^{\mu}} \tag{2.9}$$

where B is given by Eq. (2.6) and the constant A_ε is chosen to ensure normalization of $\psi_\varepsilon(t)$. The correspondence between (2.8) and (2.9) is established by taking the parameter ε to be of the same order as the parameter p. The choice (2.9) for the perturbed waiting time distribution has the advantage of considerably simplifying the calculations.

3. STATIONARITY AND NONSTATIONARITY

The two continuous-time random walk (CTRW) models introduced by Zumofen and Klafter, the velocity model (VM) and jump model[4], and the master equation theory developed by Trefán *et al.*[1] all reproduce the same situation of enhanced anomalous diffusion; this means that they are described by a probability distribution $P_0(x, t)$ whose second moment $\langle x^2(t)\rangle_0$ has the same anomalous dependence on time

$$\langle x^2(t)\rangle \simeq t^{2H}, \qquad t \gg 1, \quad H > 1/2 \tag{3.1}$$

Assuming $\psi(t)$ defined by Eq. (2.4) with $2 < \mu < 3$ as the distribution of sojourn times in each of the two states of the velocity of the particle, one obtains

$$2H = 4 - \mu \tag{3.2}$$

In the long-time regime, the probability distribution $P_0(x, t)$ of the unperturbed case, which has been studied, for example, in ref. 4, coincides with a Lévy distribution of index $\alpha = \mu - 1$. This distribution has long tails with an inverse-power-law behavior, producing the anomalous time dependence of $\langle x^2(t)\rangle_0$.

The form of the probability distribution for random walk theories depends on whether we take the system in a stationary or nonstationary state condition, as pointed out in ref. 9. We have a nonstationary state condition if the first motion event starts at $t = 0$, when observation begins; in terms of the map generating anomalous diffusion, the initial iteration is chosen as an injection step into a laminar phase. For sufficiently long times, transitions occur at the constant rate $\langle t\rangle$, defined as the first moment of the waiting time distribution $\psi(t)$; this is obviously not true for short times, where the time homogeneity of the process is lost.

If our system is in a stationary state, motion events should occur at the rate $\langle t\rangle$ at all times. This means that one has to take into account the possibility that the first observed motion event could have started some time before $t = 0$, and consequently one has to treat the first motion event differently from subsequent events.

The main difference between the two approaches lies in the shape of the distribution $P(x, t)$: in the stationary case, it has a peak at each of the two positions $x = \pm t$, corresponding to particles that were moving with velocity $\xi = \pm 1$ at $t = 0$ and kept the same velocity up to time t. However, the exponent H describing the anomalous long-time behavior of $\langle x^2(t)\rangle_0$ is the same in the stationary or nonstationary state condition. The different models, in their stationary or nonstationary version, basically give an equally good description of anomalous diffusion. Nevertheless, the VM in its stationary version has a feature which makes it the most satisfactory theory: as pointed out in ref. 10, the velocity autocorrelation function of the model coincides with the one obtained using renewal theory, namely (1.9).

Let us refer, from now on, to the perturbed case. When considering stationary conditions, it is clear that the probability distribution $P(x, t)$ will display two nonsymmetric peaks at $x = \pm t$. However, as shown in ref. 1, the long-time behavior of $\langle x(t)\rangle$ is the same as the one corresponding to the nonstationary condition.

In the perturbed case the process has an intrinsic time scale, $1/\varepsilon$. The behavior of $\langle x(t)\rangle$ radically changes if we consider times $t \ll 1/\varepsilon$ or times $t \gg 1/\varepsilon$. As far as the dependence of $\langle x(t)\rangle$ on ε and t is concerned, we have for the models discussed in ref. 4

$$\text{(i)} \qquad \langle x(t)\rangle \propto \varepsilon \langle x^2(t)\rangle_0 \propto \varepsilon t^{4-\mu}, \qquad 1 \ll t \ll 1/\varepsilon \tag{3.3}$$

$$\text{(ii)} \qquad \langle x(t)\rangle \simeq Ut \propto \varepsilon^{\mu-2} t, \qquad t \gg 1/\varepsilon \tag{3.4}$$

where U is the dimensionless mean velocity given by

$$U = \frac{\langle t_1\rangle - \langle t_{-1}\rangle}{\langle t_1\rangle + \langle t_{-1}\rangle} \tag{3.5}$$

and $\langle t_1\rangle$ and $\langle t_{-1}\rangle$ are the mean sojourn times in the right and left laminar regions, respectively.

The behavior of $\langle x(t)\rangle$ has been discussed in detail in ref. 1. In case (i), $\langle x(t)\rangle$, being proportional to the unperturbed second moment $\langle x^2(t)\rangle_0$, follows the GK prescription in its generalized form (1.3). Case (i) describes a situation of anomalous transport in which the conductivity increases as a function of time, as already pointed out in Section 1. Thus, the GK prescription cannot be valid at all times, since this would imply a diverging conductivity. In fact, the system slowly moves from this condition into regime (ii), which is described by a time-independent conductivity.

Note that all the theoretical models mentioned earlier result in the properties described by (3.3) and (3.4). The VM in its stationary version is an exception: it reproduces only the long-time regime (ii) of the process, namely

$$\langle x(t)\rangle = Ut \tag{3.6}$$

for all values of t. Thus, it displays the same behavior both at times longer or shorter than the characteristic perturbation time $1/\varepsilon$. This is so because the system is always in a state of equilibrium, and (3.6) is exactly what one would expect from a system in a stationary state. This confirms that the VM is the most reliable stationary model in the sense noted earlier.

4. SHAPE OF THE DISTRIBUTION

In this section we focus our attention on the probability distribution $P(x, t)$ of the diffusing variable x of the dynamical system under study. It is important to remark that to leading order the theoretical expression for $P(x, t)$ does not depend on which among the models quoted in Section 3 we consider. However, we refer our calculations to only one of them, the VM, assumed to be in the nonstationary condition (nonstationary in the sense illustrated in Section 3). The inverse Fourier–Laplace transform of its probability distribution reads[4]

$$\hat{P}(k, s) = \frac{\hat{\Psi}(k, s)}{1 - \hat{\psi}(k, s)} \tag{4.1}$$

where $\hat{\psi}(k, s)$ is the Fourier–Laplace transform of $\psi(x, t)$, defined as the probability distribution for moving a distance x in time t in a single motion event,

$$\psi(x, t) = \tfrac{1}{2}\delta(x - t)\,\psi_1(t) + \tfrac{1}{2}\delta(x + t)\,\psi_{-1}(t) \tag{4.2}$$

$\hat{\Psi}(k, s)$ is the Fourier–Laplace transform of $\Psi(x, t)$, defined as the probability to pass to location x at time t in a single motion event,

$$\Psi(x, t) = \tfrac{1}{2}\delta(x - t)\int_t^{\infty} dt'\psi_1(t') + \tfrac{1}{2}\delta(x + t)\int_t^{\infty} dt'\psi_{-1}(t') \tag{4.3}$$

and $\psi_1(t)$ and $\psi_{-1}(t)$ are the waiting time distributions in the right and left laminar regions and are defined by Eqs. (2.4) and (2.9), respectively. So, $P(x, t)$ reads

$$P(x, t) = \int_{a - i\infty}^{a + i\infty} ds\, e^{st} \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{-ikx}\hat{P}(k, s) \tag{4.4}$$

with $\hat{P}(k, s)$ given by Eq. (4.1). It is meaningful to study $P(x, t)$ for long times, because it is only in this limit that one can find physical features independent of the details of the model studied but pertaining to all the models generating the same inverse power law for $\psi(t)$. This implies we have to consider $\hat{P}(k, s)$ for small values of the Laplace variable s,[11, 12] and of course in the limit of small perturbation intensity ε.

The Fourier–Laplace inversion of Eq. (4.1) cannot in general be carried out analytically. However, for small values of s and ε, one can use a simplified expression for $\hat{P}(k, s)$, derived in an analogous way as that leading to the Lévy form of the probability distribution in the unperturbed case.[4, 11, 12] In particular, in the Appendix we show that, for times corresponding to the IC and CC regimes introduced in Section 3, the integrand of (4.4) can be approximated by an expression of the form

$$\hat{P}(k, s) \simeq \frac{1}{s - K(k)} \tag{4.5}$$

with $K(k)$ a particular function of k. This makes the Laplace inversion trivial, so that by inserting (4.5) into (4.4) we get

$$P(x, t) \simeq \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} e^{K(k)\, t} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \tilde{P}(k, t) \tag{4.6}$$

Let us first treat the intermediate IC regime, whose transport properties are described by Eq. (3.4). We are considering times $t \gg 1$ but such that $\varepsilon t \ll 1$. In practice, the results relative to this regime are obtained by expanding to first order in ε the factor $e^{-\varepsilon t}$ in the perturbed waiting time distribution $\psi_\varepsilon(t)$, (2.9). As shown in the Appendix, such a first-order expansion leads to the following expression for $K(k)$:

$$K(k) \simeq -\frac{c}{2\langle t \rangle} [(-ik)^{\mu-1} + (ik)^{\mu-1} + (\mu - 1)\, \varepsilon (ik)^{\mu-2}] \tag{4.7}$$

where $\langle t \rangle$ is the first moment of $\psi(t)$,

$$\langle t \rangle = \int_0^{\infty} t\psi(t)\, dt = \frac{B}{\mu - 2} \tag{4.8}$$

and c is defined by

$$c = B^{\mu-1} \Gamma(2 - \mu) \tag{4.9}$$

In the limit of large $|x|$, the Fourier inversion of $\tilde{P}(k, t) = e^{K(k)t}$ can be carried out analytically, with the technique illustrated in ref. 11. The resulting expression, which is in the form of an asymptotic power series, includes terms that are symmetric for $x \to -x$ and therefore do not contribute to $\langle x(t) \rangle$, and odd terms in which we are particularly interested, since we are examining the properties of transport. The leading symmetric term is

$$P(x, t)_{\text{sym}} \simeq q \frac{t}{|x|^{\mu}}, \qquad |x| \gg t^{1/(\mu-1)} \tag{4.10}$$

while the leading nonsymmetric term, which is present only for $x < 0$, is

$$P(x, t)_{\text{asym}} \simeq -q \frac{\varepsilon t}{|x|^{\mu-1}}, \qquad |x| \gg t^{1/(\mu-1)}, \quad x < 0 \tag{4.11}$$

where q is defined by

$$q = -\frac{c}{2\langle t \rangle} \frac{\Gamma(\mu) \sin(\pi\mu)}{\pi} \tag{4.12}$$

We thus see that in this part of the process the perturbation has the effect of lowering the left tail of $P(x, t)$, subtracting (4.11) from its unperturbed value [see Eq. (4.23)]. Let us see how this leads to a $\langle x(t) \rangle$ which has the t and ε dependence predicted by (3.3). Supposing that the expression for $P(x, t)$ given by (4.10), (4.11) is valid for every $x < t$ [since the δ-function condition in the expression for $\psi(x, t)$ implies that $P(x > t, t) = 0$], we can calculate an approximate value of $\langle x(t) \rangle$

$$\langle x(t) \rangle \simeq \int_{-t}^{t} x P(x, t)\, dx \simeq -q\varepsilon t \int_{-t}^{0} \frac{x}{|x|^{\mu-1}}\, dx \propto \varepsilon t^{4-\mu}, \qquad t \ll 1/\varepsilon \tag{4.13}$$

which coincides with (3.3). We thus see that, for times smaller than the characteristic time of the perturbation, the asymmetry of the tails produces the anomalous dependence on time of $\langle x(t) \rangle$.

Let us now study $P(x, t)$ for times $t \gg 1/\varepsilon$, when the properties of transport are described by Eq. (3.4) of Section 3. As shown in the Appendix in this case Eq. (4.1) can again be approximated by an expression of the form (4.5), with $K(k)$ given by

$$K(k) \simeq -\frac{c}{2\langle t \rangle} [(-ik)^{\mu-1} + (\varepsilon + ik)^{\mu-1} - \varepsilon^{\mu-1}] \tag{4.14}$$

which for large times can be further simplified as

$$K(k) \simeq -ik \frac{(\mu-1)\, c}{2\langle t \rangle} \varepsilon^{\mu-2} - \frac{c}{2\langle t \rangle} (-ik)^{\mu-1} \tag{4.15}$$

In this last case, the characteristic function $\tilde{P}(k, t) = e^{K(k)t}$ is thus of the form

$$\tilde{P}(k, t) \simeq \exp\{ikUt - b\,|k|^{\alpha} t[1 + ig\omega(\alpha)\,\mathrm{sgn}(k)]\} \tag{4.16}$$

where

$$\alpha = \mu - 1 \tag{4.17}$$

$$\omega(\alpha) = \mathrm{tg}\left(\frac{\pi\alpha}{2}\right) \tag{4.18}$$

$$g = -1 \tag{4.19}$$

$$b = \frac{c}{2\langle t \rangle} \cos\left(\frac{\pi\alpha}{2}\right) \tag{4.20}$$

and

$$U = -\frac{c}{2\langle t \rangle} (\mu - 1)\, \varepsilon^{\mu - 2} \tag{4.21}$$

Notice that U is the mean velocity appearing in Eqs. (3.4) and (3.5), calculated to the lowest order in ε. Now, Eq. (4.16) with $\omega(\alpha) = tg(\pi\alpha/2)$ and the generic real parameters $U, b \geqslant 0$, and $-1 \leqslant g \leqslant 1$ is the most general form of a Lévy stable distribution.[13] The parameter U represents the velocity of translation of the distribution; b is a scale factor which could be reabsorbed by a dilation of x. The parameter g is related to the degree of asymmetry of $P(x, t)$; it is easy to see that for $g = 0$, $P(x, t)$ is symmetric, while for $g = \pm 1$, $P(x, t)$ reaches the maximum possible asymmetry. α is the index of the Lévy distribution and determines its large-$|x|$ behavior. We thus see that, in the long-time regime $t \gg 1/\varepsilon$, the process under study reduces to the stationary asymmetric Lévy process (4.16).

Let us remark that the characteristic function $\tilde{P}_0(k, t)$ relative to the unperturbed case is given by (4.16) with $U = 0$, $g = 0$, and a value of b twice that of Eq. (4.20). Notice that the large-$|x|$ behavior of the unperturbed distribution is

$$P_0(x, t) \simeq q \frac{t}{|x|^{\mu}}, \qquad |x| \gg t^{1/(\mu - 1)} \tag{4.22}$$

with q still given by (4.12).

Both, the probability distributions relative to the unperturbed case and the perturbed case in the long-time stationary regime are α-stable Lévy distributions, and so exhibit the characteristic scaling property (with $U=0$ for the unperturbed case)

$$P(x, t) = \frac{1}{t^{1/(\mu-1)}} F\left(\frac{x-Ut}{t^{1/(\mu-1)}}\right) \tag{4.23}$$

which is equivalent to the condition for the characteristic function

$$\tilde{P}(k, t) = e^{ikUt}\tilde{F}(kt^{1/(\mu-1)}) \tag{4.24}$$

Let us also notice that the characteristic function $\tilde{P}(k, t)$ of the intermediate nonstationary regime, relative to times $1 \ll t \ll 1/\varepsilon$, does not have the scaling property (4.23).

The transport property $\langle x(t)\rangle$ has contributions from both the drift term, namely the velocity U of the distribution, and the asymmetric tail of the distribution. It is easy to see that in the long-time region the leading contribution is the one that comes from the drift

$$\langle x(t)\rangle_{\text{drift}} = Ut \tag{4.25}$$

which coincides in fact with Eq. (3.4).

Let us see what changes in this description if we consider the VM in the stationary condition; we recall that the model has the interesting feature of describing only the steady-state condition $\langle x(t)\rangle \simeq Ut$, as an acceptable stationary model should do. The probability distribution of the stationary VM reads, generalizing the treatment of ref. 9 to the case with perturbation,

$$\begin{aligned} P(x, t) &= L^{-1}F^{-1}\left\{\frac{2}{\langle t_1\rangle + \langle t_{-1}\rangle}\frac{[\hat{\Psi}(k, s)]^2}{1-\hat{\psi}(k, s)}\right\} + H(x, t) \\ &\equiv R(x, t) + H(x, t) \end{aligned} \tag{4.26}$$

where $L^{-1}F^{-1}$ indicates inverse Fourier–Laplace inversion; $\langle t_1\rangle$ and $\langle t_{-1}\rangle$ are the first moments of $\psi(t)$ and $\psi_\varepsilon(t)$, respectively; $\hat{\psi}(k, s)$, $\hat{\Psi}(k, s)$ are the Fourier–Laplace transforms of Eqs. (4.2), (4.3); $H(x, t)$ is the probability to pass location x at time t during the first motion event, and has the expression

$$\begin{aligned} H(x, t) = &\frac{1}{\langle t_1\rangle + \langle t_{-1}\rangle} \\ &\times\left[\delta(x-t)\int_t^\infty dt'(t'-t)\,\psi(t) + \delta(x+t)\int_t^\infty dt'(t'-t)\,\psi_\varepsilon(t)\right] \end{aligned} \tag{4.27}$$

In the limit of long times, the term $R(x, t)$ has the same qualitative behavior as the complete probability distribution $P(x, t)$ of the nonstationary VM [it is easy to verify this using the limiting expressions for $\hat{\psi}(k, s)$, $\hat{\Psi}(k, s)$ given in the Appendix].

The term $H(x, t)$ contributes only for $x = \pm t$ and it is responsible for the two asymmetric peaks, which to leading order behave as

$$x = t: \qquad H(x, t) \simeq \frac{1}{2}\left(\frac{t}{B}\right)^{2-\mu} \tag{4.28}$$

$$x = -t: \qquad H(x, t) \simeq \frac{1}{2}\left(\frac{t}{B}\right)^{2-\mu} - U, \qquad t \ll \frac{1}{\varepsilon} \tag{4.29}$$

$$H(x, t) \simeq \frac{e^{-\varepsilon t}}{\varepsilon t^{\mu-1}} \qquad t \gg \frac{1}{\varepsilon} \tag{4.30}$$

It is the interplay of $R(x, t)$ [the central region of $P(x, t)$] and $H(x, t)$ (the peaks) that produces the peculiar features of the stationary VM described in Section 3. What happens is that the contribution of $R(x, t)$ to $\langle x(t) \rangle$ cancels with part of the contribution of $H(x, t)$. The remaining part of the contribution of the asymmetry of the peaks is exactly $\langle x(t) \rangle = Ut$.

We have seen that perturbation (2.9) induces a transition from the unperturbed α-stable Lévy process to another Lévy process of the same index α, with maximum asymmetry. The Lévy property remains, even if the scale has been introduced. Let us consider, however, a perturbation defined in the following way:

$$\psi_1(t) = A_{\varepsilon_r} \frac{e^{-\varepsilon_r t}}{(B+t)^\mu} \tag{4.31}$$

$$\psi_{-1}(t) = A_{\varepsilon_l} \frac{e^{-\varepsilon_l t}}{(B+t)^\mu} \tag{4.32}$$

where the constants A_{ε_r} and A_{ε_l} are chosen to ensure normalization.

It is easy to verify, following the derivation carried out in ref. 1, that the response $\langle x(t) \rangle$ has an analogous behavior to that relative to perturbation (2.9). Taking ε_l, ε_r to be of the same order, we can distinguish two time regimes:

$$\langle x(t) \rangle \propto (\varepsilon_l - \varepsilon_r)\langle x^2(t) \rangle_0 \propto (\varepsilon_l - \varepsilon_r)\, t^{4-\mu}, \qquad t \ll 1/\varepsilon_l, \quad t \ll 1/\varepsilon_r \tag{4.33}$$

where the response also follows the Green–Kubo relation (1.3), and

$$\langle x(t) \rangle \simeq \tilde{U} t \propto (\varepsilon_l^{\mu-2} - \varepsilon_r^{\mu-2})\, t, \qquad t \gg 1/\varepsilon_l, \quad t \gg 1/\varepsilon_r \tag{4.34}$$

which is the stationary regime of constant conductivity $\tilde{U}$. $\tilde{U}$ is given by the general equation (3.5), with $\langle t_1 \rangle$, and $\langle t_{-1} \rangle$ the first moments of $\psi_1(t)$ and $\psi_{-1}(t)$ defined by Eqs. (4.31) and (4.32), respectively.

The fact that in this case the power-law tails of, both $\psi_1(t)$ and $\psi_{-1}(t)$ are cut by an exponential factor at large times implies that the stationary distribution $P(x, t)$ has a completely different behavior from the Lévy asymmetric one we discussed before. As shown in the Appendix, at times $t \gg 1/\varepsilon_l$, $t \gg 1/\varepsilon_r$, the Fourier–Laplace transform of $P(x, t)$ has the form (4.5), with $K(k)$ given by

$$K(k) \simeq -ik \frac{(\mu-1)\, c}{2\langle t \rangle} (\varepsilon_l^{\mu-2} - \varepsilon_r^{\mu-2}) + k^2 \frac{(\mu-1)(\mu-2)\, c}{4\langle t \rangle} (\varepsilon_l^{\mu-3} + \varepsilon_r^{\mu-3}) \tag{4.35}$$

The characteristic function $\tilde{P}(k, t) = e^{K(k)t}$ is thus the inverse Fourier transform of the Gaussian distribution

$$P(x, t) \simeq \left(\frac{1}{4\pi \tilde{D} t} \right)^{1/2} \exp\left[-\frac{(x - \tilde{U}t)^2}{4\tilde{D}t} \right] \tag{4.36}$$

with

$$\tilde{U} = -\frac{(\mu-1)c}{2\langle t \rangle} (\varepsilon_l^{\mu-2} - \varepsilon_r^{\mu-2}) \tag{4.37}$$

$$\tilde{D} = -\frac{(\mu-1)(\mu-2)\, c}{4\langle t \rangle} (\varepsilon_l^{\mu-3} + \varepsilon_r^{\mu-3}) \tag{4.38}$$

Notice that the diffusion coefficient $\tilde{D}$ diverges as ε_l and ε_r go to zero. This evidently depends on the fact that in the unperturbed case diffusion was anomalous, due to the inverse-power-law behavior of $\psi(t)$.

5. STATISTICAL NATURE OF THE IC AND CC REGIMES

This section illustrates the nature of the IC and CC regimes with the help of a numerical treatment of the Geisel map and the numerical Fourier inversion of the theoretical expressions for $\tilde{P}(k, t)$ given in Section 4.

Figure 2 shows the time evolution of $\langle x(t) \rangle$ obtained by a numerical treatment of the perturbed Geisel map defined in Sec. 2, with a value of $\mu = 2.25$ and a perturbation parameter $\varepsilon = 0.005$. The purpose of the figure is to illustrate the transition from the IC to the CC regime, theoretically described by Eqs. (3.3) and (3.4), respectively. The numerical data are

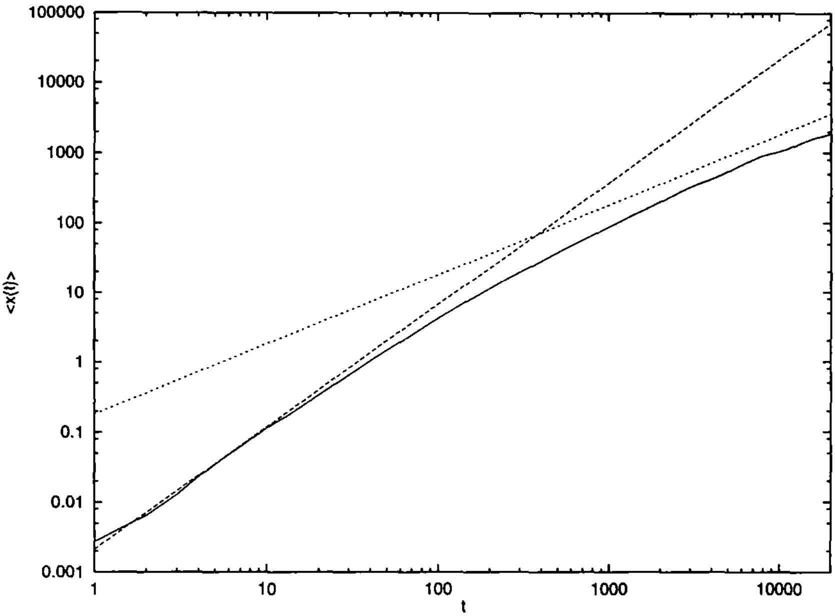

Fig. 2. Evolution of $\langle x(t)\rangle$ up to times $t = 20{,}000$. The solid line is the result of numerical calculations obtained by the direct realization of the perturbed Geisel map. The dashed lines represent the theoretical prediction of the VM: the long-dashed line is obtained from Eq. (5.1) calculated at times $1 \ll t \ll 1/\varepsilon$; the short-dashed line is obtained from Eq. (5.1) calculated at times $t \gg 1/\varepsilon$. The values used for μ, ε are $\mu = 2.25$ and $\varepsilon = 0.005$.

compared with the theoretical prediction of the VM in its nonstationary condition, namely, with

$$\langle x(t)\rangle = L^{-1}\left\{-i\left[\frac{\partial \hat{P}(k,s)}{\partial k}\right]_{k=0}\right\} \tag{5.1}$$

where $\hat{P}(k, s)$ is given by Eq. (4.1). With the help of a log-log plot it is clearly shown that the numerical treatment and the theory agree with one another, and show an evident change of regime at times of the order of $1/\varepsilon$.

In Fig. 3, for the same parameters $\mu = 2.25$ and $\varepsilon = 0.005$ as Fig. 2, we illustrate the corresponding change in the shape of the distribution. The numerical results are shown only up to the transition time $1/\varepsilon$, because after this time they become increasingly inaccurate. We note that the numerical results are obtained by applying abruptly the perturbation to the system at time $t = 0$. At this time, the system is set into a state corresponding to the stationary unperturbed distribution of the velocity. This special

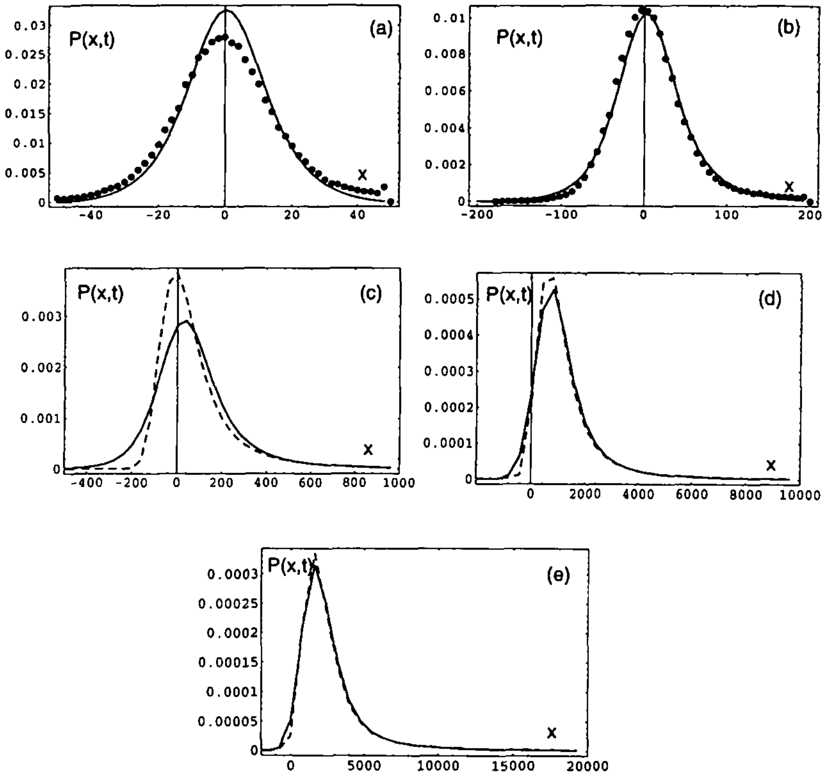

Fig. 3. Evolution of $P(x, t)$ at times (a) $t = 50$, (b) $t = 200$, (c) $t = 1000$, (d) $t = 10{,}000$, (e) $t = 20{,}000$. The dots show the results of numerical calculations obtained by the direct realization of the perturbed Geisel map; the solid lines are the results of the numerical inverse Fourier transformation using kernel (4.7) for (a) and (4.14) for (b)–(e). The dashed lines of (c)–(e) are the results of the numerical inverse Fourier transformation of (4.16). The values used for μ, ε are $\mu = 2.25$ and $\varepsilon = 0.005$.

condition has the effect of yielding the distribution peaks at the positions $x = \pm t$. These peaks have the same origin as those discussed at the end of the preceding section, which are also widely discussed by Klafter and Zumofen.[9] For the sake of simplicity the theoretical prediction was obtained by adopting an initial condition corresponding to the nonstationary condition with perturbation, described in Sec. 3. It is possible to prove, however, that the choice of initial conditions identical to that of the numerical experiment would not affect the distribution between the two peaks. Consequently, the marked discrepancy between theory and numerical experiment in Fig. 3a must be due to different reasons. Probably this is in part due to the fact that the theoretical prediction refers to long times, and therefore is totally independent of the nature of the generator of the inverse-power-law adopted, whereas the numerical experiment in the short-time region is still influenced by the details of this generator.

The wide region $t \gg 1/\varepsilon$ is illustrated only making use of the Fourier transform of $\tilde{P}(k, t) = e^{K(k)t}$ with $K(k)$ given by Eqs. (4.14) and (4.15), and serves very well the purpose of illustrating the transition from the regime of reshaping of the distribution to that of the asymmetric Lévy process.

6. CONCLUDING REMARKS

We summarize the most significant results of this paper:

(i) The effect of an abrupt perturbation of a process of anomalous diffusion is that of creating a long-time regime of transition to a new stationary regime. The time duration of this process is not independent of the intensity of the perturbation, as in the case of normal diffusion. Rather, the smaller is the perturbation intensity, the longer is the time duration of the transition regime. The breakdown of the linear response regime, the GK regime, is closely related to the fact that the process of readjustment of the variable velocity does not have a finite time scale. Consequently, the time scale for the process of x-redistribution depends on the time scale at which the inverse-power-law distribution of waiting times is truncated. This means that in the long-time regime the response depends on the properties of the perturbed waiting time distributions $\psi_1(t)$ and $\psi_{-1}(t)$ rather than on the unperturbed correlation function $\Phi_\xi(t)$ as in the conventional linear response treatment. In other words, the transition to the final stationary regime cannot be described by a linear response treatment. The transition from the initial symmetric Lévy process to the final asymmetric Lévy process implies a breakdown of the conventional predictions of ordinary statistical mechanics.

(ii) The statistical nature of the new stationary regime depends on whether or not both waiting time distributions are truncated at a given time of the order of $1/\varepsilon$. In the former case the new waiting time distribution would be Gaussian. More interesting is the case analyzed in this paper, corresponding to truncating only one of the two waiting time distributions. The new stationary process turns out to be an asymmetric Lévy process. This result is in line with the well-known fact that the Lévy processes are stable, as are the Gaussian ones.

(iii) When both waiting time distributions are truncated and an asymmetric Lévy process is generated, the process of transport in one direction also depends on the dynamics of the tails of the distribution. This is reminiscent of results found several years ago by other authors.[14, 15]

However, in addition to being devoted to a case of superdiffusion, rather than subdiffusion as in refs. 14 and 15, our treatment rests on a specific dynamic derivation and establishes a comparison with the conventional Green–Kubo regime.

APPENDIX

We recall that the distribution of waiting times in the two states of the velocity $\xi = \pm 1$ in the unperturbed case is

$$\psi(t) = \frac{(\mu - 1)\, B^{\mu - 1}}{(B + t)^{\mu}} \tag{A.1}$$

with B a parameter given by (2.6) and $2 < \mu < 3$.

Let us first consider the case of perturbation (2.9), which is applied by modifying the waiting-time distribution in the state $\xi = -1$ of the velocity and is defined by

$$\psi_1(t) = \psi(t) \tag{A.2}$$

$$\psi_{-1}(t) = \frac{e^{\varepsilon t}\psi(t)}{\hat{\psi}(\varepsilon)} \tag{A.3}$$

where $\psi(t)$ is given by (A.1) and the function $\hat{\psi}$ denotes the Laplace transform of $\psi(t)$. In the following, the Fourier–Laplace transform of a function $f(x, t)$ will be indicated by $\hat{f}(k, s)$. Let us now consider the quantities $\hat{\psi}(k, s)$ and $\hat{\Psi}(k, s)$, which are necessary to compute the probability distribution $P(x, t)$:

$$P(x, t) = \int_{a - i\infty}^{a + i\infty} ds\, e^{st} \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{ikx} \frac{\hat{\Psi}(k, s)}{1 - \hat{\psi}(k, s)} \tag{A.4}$$

From the definitions of $\psi(x, t)$ and $\Psi(x, t)$ in Eqs. (4.2), (4.3) it follows that, for this choice of the perturbation,

$$\hat{\psi}(k, s) = \frac{1}{2}\,\hat{\psi}(s - ik) + \frac{1}{2}\,\frac{\hat{\psi}(s + ik + \varepsilon)}{\hat{\psi}(\varepsilon)} \tag{A.5}$$

and

$$\hat{\Psi}(k, s) = \frac{1}{2} \frac{1}{s - ik} [1 - \hat{\psi}(s - ik)] + \frac{1}{2} \frac{1}{s + ik} \left[1 - \frac{\hat{\psi}(s + ik + \varepsilon)}{\hat{\psi}(\varepsilon)} \right] \tag{A.6}$$

so that everything is formally expressed in terms of the Laplace transform of $\psi(t)$.

We are interested in evaluating $P(x, t)$ for large times and small perturbation intensities. So, expressions (A.5) and (A.6) are to be substituted into Eq. (A.4) and have to be evaluated in the limit of small s, ε. The leading contribution to the integral (4.4) comes from the region of small k's, because the denominator $1 - \hat{\psi}(k, s)$ goes to zero when $s, k, \varepsilon \to 0$, so that we have to consider $\hat{\psi}$ in equals (4.1) for small values of the complex arguments $s - ik$, $s + ik + \varepsilon$.

The first terms of the small-s expansion of $\hat{\psi}(s)$ are, for $2 < \mu < 3$,

$$\hat{\psi}(s) \simeq 1 - \langle t \rangle - cs^{\alpha} \tag{A.7}$$

where $\langle t \rangle$ and c are defined in Eqs. (4.8) and (4.9) respectively, and α is defined by

$$\alpha = \mu - 1 \tag{A.8}$$

Using the fact that the complex arguments of $\hat{\psi}(q)$ in Eqs. (A.5) and (A.6) have positive real part, we can extend the expansion (A.7) to $q = s - ik$, $q = s + ik + \varepsilon$.

The numerator $\hat{\Psi}(k, s)$ in Eq. (A.4) has a finite value at $\varepsilon = s = k = 0$, so that it can be approximated, to lowest order, by

$$\hat{\Psi}(k, s) \simeq \langle t \rangle \tag{A.9}$$

The expression for $\hat{\psi}(k, s)$ corresponding to the IC case is obtained expanding (A.5) to first order in ε, which amounts to expanding the factor $\exp(-\varepsilon t)$ in the perturbed waiting time distribution $\psi_{\varepsilon}(t)$ to first order in εt. Moreover, we consider the expression for $\hat{\psi}(k, s)$ in the range $s \ll k$. As discussed in refs. 11 and 12, this is a reasonable approximation in most of the (s, k) domain. Anyway, one should remember that the expression obtained cannot be directly used in the calculation of quantities like $\langle x(t) \rangle$

and $\langle x^2(t) \rangle$, which require a knowledge of the derivatives of $\hat{P}(k, s)$ for $k = 0$. The result is:

$$\hat{\psi}(k, s) \simeq 1 - \langle t \rangle s - \frac{c}{2} [(-ik)^\alpha + (ik)^\alpha + \alpha\varepsilon(ik)^{\alpha-1}] \tag{A.10}$$

Notice that Eq. (A.10) evaluated at $\varepsilon = 0$ is identical to the approximate expression (14) of ref. 12 for the unperturbed $\hat{\psi}(k, s)$.

The expression for $\hat{\psi}(k, s)$ corresponding to the CC case is obtained expanding (A.5) for $s \ll \varepsilon$, which amounts to considering the factor $\exp(-\varepsilon t)$ in the perturbed waiting time distribution $\psi_\varepsilon(t)$ for large values of εt:

$$\hat{\psi}(k, s) \simeq 1 - \langle t \rangle s - \frac{c}{2} [(-ik)^\alpha + (\varepsilon + ik)^\alpha - \varepsilon^\alpha] \tag{A.11}$$

It is the very small values of k, $k \ll \varepsilon$, that give the dominant contribution to the integral (A.4). Approximating (A.11) for $k \ll \varepsilon$, we obtain the expression for $\hat{\psi}(k, s)$ corresponding to the stationary Lévy asymmetric regime

$$\hat{\psi}(k, s) \simeq 1 - \langle t \rangle s - \frac{c}{2} (-ik)^\alpha - \frac{\alpha c}{2} \varepsilon^{\alpha-1}(ik) \tag{A.12}$$

Substituting Eqs. (A.9)–(A.12) into (4.1), it is straightforward to verify that the resulting expressions for $\hat{P}(k, s)$ always have the form (4.5), with $K(k)$ given by Eqs. (4.7), (4.14), and (4.15) of the text.

Let us now consider the case of the perturbation defined by Eqs. (4.31), (4.32), which is applied by modifying the waiting time distributions in both states $\xi = +1$ and $\xi = -1$ of the velocity in a way analogous to (2.9):

$$\psi_1(t) = \frac{e^{-\varepsilon_r t} \psi(t)}{\hat{\psi}(\varepsilon_r)} \tag{A.13}$$

$$\psi_{-1}(t) = \frac{e^{-\varepsilon_l t} \psi(t)}{\hat{\psi}(\varepsilon_l)} \tag{A.14}$$

Let us consider the quantities $\hat{\psi}(k, s)$ and $\hat{\Psi}(k, s)$, which in this case have the formal expressions in terms of the Laplace transform $\hat{\psi}$ of the unperturbed waiting time distribution $\psi(t)$:

$$\hat{\psi}(k, s) = \frac{1}{2} \frac{\hat{\Psi}(s - ik + \varepsilon_r)}{\hat{\psi}(\varepsilon_r)} + \frac{1}{2} \frac{\hat{\psi}(s + ik + \varepsilon_l)}{\hat{\psi}(\varepsilon_l)} \tag{A.15}$$

$$\hat{\Psi}(k, s) = \frac{1}{2}\frac{1}{s - ik}\left[1 - \frac{\hat{\psi}(s - ik + \varepsilon_r)}{\hat{\psi}(\varepsilon_r)}\right] + \frac{1}{2}\frac{1}{s + ik}\left[1 - \frac{\hat{\psi}(s + ik + \varepsilon_l)}{\hat{\psi}(\varepsilon_l)}\right] \tag{A.16}$$

With the same kinds of approximations used in the case of perturbation (A.2), (A.3) we derive, to lowest significant order,

$$\hat{\Psi}(k, s) \simeq \langle t \rangle \tag{A.17}$$

for both the IC and CC regimes. The expression for $\hat{\psi}(k, s)$ is

$$\hat{\psi}(k, s) \simeq 1 - \langle t \rangle\, s - \frac{c}{2}\left[(-ik)^\alpha + (ik)^\alpha + \alpha(\varepsilon_l - \varepsilon_r)(ik)^{\alpha - 1}\right] \tag{A.18}$$

in the IC case, and

$$\hat{\psi}(k, s) \simeq 1 - \langle t \rangle\, s - \frac{c}{2}\left[(\varepsilon_r - ik)^\alpha - \varepsilon_r^\alpha + (\varepsilon_l + ik)^\alpha - \varepsilon_l^\alpha\right] \tag{A.19}$$

in the CC case. For $k \ll \varepsilon$ this last relation becomes

$$\hat{\psi}(k, s) \simeq 1 - \langle t \rangle\, s + \frac{(\mu - 1)(\mu - 2)c}{4}\,(\varepsilon_l^{\alpha - 2} + \varepsilon_r^{\alpha - 2})\,k^2 - \frac{\alpha c}{2}\,(\varepsilon_l^{\alpha - 1} - \varepsilon_r^{\alpha - 1})(ik) \tag{A.20}$$

Substituting Eqs. (A.17) and (A.20) into (4.1), one finds that the resulting expression for $\hat{P}(k, s)$ has the form (4.5), with $K(k)$ given by Eq. (4.35) of the text.

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